B. J. Day

THE ANALYST;

OR,

MATHEMATICAL MUSEUM.

CONTAINING

NEW ELUCIDATIONS, DISCOVERIES AND IMPROVEMENTS,

IN VARIOUS BRANCHES OF THE

MATHEMATICS,

WITH COLLECTIONS OF QUESTIONS

PROPOSED AND RESOLVED

BY INGENIOUS CORRESPONDENTS.

PART I. No. III.

UTILE DULCI.

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THE ANALYST;

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ARTICLE VII.

SOLUTIONS TO THE QUESTIONS PROPOSED IN ARTICLE VI.

QUESTION I.

Solved by the Editor.

THE magnitudes of the quantities sought are evidently as their weights directly, and specific gravities inversely; but their weights are as 4 to 3.3, and their specific gravities as 1 to .825, therefore their magnitudes are as $4\times.825=3.3$ to $3.3\times1=3.3$, that is, their magnitudes are equal. Again, the specific gravity of a mixture of equal quantities of water and alcohol would be $\frac{1}{2}(1+.825)=.9125$ were the ingredients to retain their magnitudes; but, by the question, the specific gravity of the mixture is .9388, which shows that the magnitude of the compound is less than that of the fluids when unmixed in the ratio of 9388 to 9125; since then the quantity or magnitude of the mixture is $31\frac{1}{2}$ gallons, we have only to say, as 9125 : 9388 : : 31.5 : 32.4078, the half of which, 16.2039 gallons, is the quantity of each ingredient required.

QUESTION II.

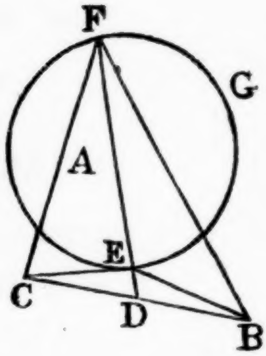
Solved by Daniel Smith, jun.

Put a=the given area=100 acres, b=diagonal=100 four-pole chains; and x, $2x$, $3x$, and $4x$, the lengths of the required sides; then the area of the triangle, bounded by $2x$, $3x$ and b, is $\frac{1}{4}(25x^2-b^2)^{\frac{1}{2}}\times(b^2-x^2)^{\frac{1}{2}}$, and the area of that of which the sides same are x, $4x$, and b, is $\frac{1}{4}(25x^2-b^2)^{\frac{1}{2}}\times(b^2-9x^2)^{\frac{1}{2}}$, the sum of which

$a=(25x^2-b^2)^{\frac{1}{2}}\times\left\{(b^2-x^2)^{\frac{1}{2}}+(b^2-9x^2)^{\frac{1}{2}}\right\}$; this equation cleared of surds, &c. produces one of the 4th degree, from which we find $x=20.50596$, consequently the required sides are 20.5, 41, 61.5, and 82, respectively.

QUESTION III.

Solved by the Proposer, John Gummere.



Let EFG be the given circle, A its centre, and B, C, the given points; join BC, and bisect it in D, and draw from D through the centre A the straight line DEF, cutting the circumference in E and F, the points required. For join BE, CE; and by Prop. A, B. II, *Playfair's Geom.* $\overline{EB}^2+\overline{EC}^2=2\overline{BD}^2+2\overline{DE}^2$; therefore, since $2\overline{BD}^2$ is constant, $\overline{EB}^2+\overline{EC}^2$ varies as $2\overline{DE}^2$; but when DE produced passes through the centre A, it is evidently the shortest line that can be drawn from D to the circumference; $\overline{EB}^2+\overline{EC}^2$ is therefore a minimum: in like manner it is proved that $\overline{FB}^2+\overline{FC}^2$ is a maximum.

QUESTION IV.

Solved by Seth Smith.

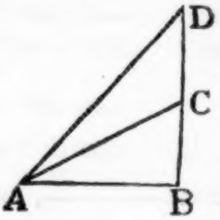
Put the sine of the required latitude $=x$, the tangent of the three-o'clock hour-arch=tangent of 45°=radius=1, and the tangent of the 5 o'clock hour-arch=tangent of $75^\circ=t$: then by spherics, as $1 : x :: 1 : x$=tangent of the angle contained between the meridian, and the 3 o'clock hour-line, and therefore the tangent of double that angle is $\frac{2x}{1-x^2}$; also, as $1 : x :: t : tx$= the tangent of the angle contained between the meridian and the 5 o'clock hour-line, whence $tx=\frac{2x}{1-x^2}$, and therefore $t=\frac{2}{1-x^2}$, from which is found $x=\sqrt{\frac{t-2}{t}}$=sine of $42^\circ\ 56'\ 29''$, the required latitude.

The same Question solved by the Editor.

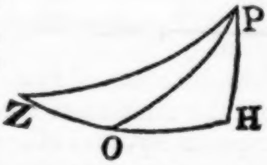
Let r=radius=tang. of 45°, t=cotang. of 75°, x=sine, and y=cosine of the latitude required. By spherics, as $r : x :: r : x$ =tangent of the three-o'clock hour-angle, and the tangent of double this angle is $\frac{2r^2x}{y^2}$: again, as $r : x :: \frac{r^2}{t}$=tangent of 75°: $\frac{rx}{t}$=tangent of the five-o'clock hour-angle, hence $\frac{rx}{t}=\frac{2r^2x}{y^2}$, from which $y^2=2rt$, and $y=\sqrt{2rt}$; or by the logarithmic tables, 2 log. y=log. 2+log. r+log. t. Operation by the tables;

Log. t=log. cotang. of 75°	= 9.4280525
Logarithm of 2	= .3010300
Logarithm of r	=10.0000000
Sum=twice the logarithm of y	=19.7290825
Log. cosine of lat. sought 42° 56′ 29″	= 9.8645412½

Another solution to the same by the Editor.



Lemma. Let BAC, CAD be equal angles, and BCD perpendicular to AB; then AB being radius, as BC, the tangent of BAC, is to CD, the difference of the tangents of BAC, BAD; so is the radius AB to AD, the secant of BAD.



Now, let HOZ be the horizon, P the pole, DH, PO, PZ the meridians of 12, 3, and 5 o'clock, PH being perpendicular to HOZ, and the angles HPO, HPZ, 15°×3, and 15°×5, or 45° and 75° respectively; also, by the question, the arc HO is equal to OZ. By spherics, as tang. of HPO, or radius, is to tang. of HPZ; so is tang. of HO to tang. of HZ; therefore as rad. is to tang. HPZ—rad. so is tang. HO to tang. HZ—tang. HO, and (by the lemma) so is rad. to secant of HZ; consequently the secant of HZ=tang. HPZ— radius =tang. 75°—1=2.7320508, whence HZ=68° 31′ 45½″, and HO=34° 15′ 52¾″, of which the tang. is .6812500, which by spherics is also the sine of PH, from which we find PH=42° 56′ 29″, the latitude required.

From the preceding analysis we may determine the latitude by the common scale of natural sines, tang. &c. thus; the extent from 45° to 75° on the tangents, being applied to the secants gives 68½°, and the tangent of half this, viz. 34¼°, applied to the sines gives 43° nearly, for the latitude sought.

QUESTION V.

Solved by Seth Smith.

Put $a=\frac{47}{188000}$=the difference between the expansions of the brass and steel by an increase of temperature of 10 degrees, and c=the circumference of a circle to the radius $\frac{1}{100}$; and since the difference of two similar arcs is the length of a similar arc to a radius equal to the difference of the radii of those arcs, therefore as $c : a :: 360^\circ : 1^\circ\ 29'\ 45.8''$, the portion of the circle required.

The same solved otherwise by Daniel Smith, jun.

From the data of the question, it is evident that with the given increase of temperature the steel will expand .00038333, &c. and the brass .00064444, &c. hence their lengths will be 6.00038333, and 6,00064444. Now as the compound strip will assume a circular form, and the radii of circles are as the lengths of similar arcs, if x=the radius of the brass arc, we have, as x : 6.00064, &c. :: x—.01=rad. of steel arc : 6.000383, &c. whence x= 229.81 inches, and the required arc=$1^\circ\ 29'\ 45.8''$.

Another solution to the same by John Gummere.

By proportion we find the length of the brass $\frac{216.0232}{36}$, and the length of the steel $\frac{216.0138}{36}$, after they are expanded by the given increase of heat. Let ab and cd represent the middles of the strips of brass and steel respectively, and aC, cC the radii of those circular strips: then since in similar sectors the arcs are as their radii, we have $ab : cd :: a\mathrm{C} : c\mathrm{C}$, and $ab—cd : ab :: a\mathrm{C}—c\mathrm{C}=ac : a\mathrm{C}=\frac{ab \times ac}{ab—cd}=\frac{21602.32}{94}$, and $\frac{21602.32}{94} \times 6.28318$ =the circumference of the circle of which the radius is aC; hence as $\frac{21602.32}{94}\times 6.28318 : \frac{21602.32}{36} :: 360^\circ : 1^\circ\ 29'\ 45.8''$, the arc required.

Solution by the proposer, Robert Patterson.

Though the compound metallic strip will, by the given increase of temperature, be indeed increased in all its dimensions, yet this absolute increase will be so very small, that it may be

safely neglected in the calculation, without producing any sensible error in the result.

Suppose a similar compound metallic strip of such a length as, by the given increase of temperature, to be bent into a complete circle; then the difference between the diameters of the two contiguous metallic circles will, from the conditions of the question, evidently$=.02$ inch, and consequently the difference of the circumferences (or lengths of the metallic strips) will $=.02\times3.141598$, &c. $=.0628316$. But the difference between the lengths of the two given strips, with the same increase of temperature $= \frac{.0232-.0138}{2\times18}=.00026\frac{1}{9}$; therefore, by the rule of proportion, say as $.0628316 : 360^{\circ} : : .00026\frac{1}{9} : 1^{\circ}\ 29'\ 45.8''$, the arc, or portion of the circle required.

The Editor thinks that the answer obtained by the preceding solutions is probably as near the truth as any that can be found: but he is decidedly of opinion that no just and unobjectionable solution can be given to this question, at least in the present state of philosophical knowledge; and doubts not that those who are the best judges of what constitutes a legitimate solution, will coincide with him in sentiment. In fact, it is entirely uncertain what interval ought to be taken between the circumferences or arcs of brass and steel, of which the lengths and difference are discoverable from the data of the question: this interval is probably not less than .01, as assumed in the preceding solutions, nor greater than .02, which has been adopted by an ingenious mathematician as a probable supposition. Mr. *Craig* observes, in a scholium to his solution, that "as the plates of brass and steel have different expansive capacities, and as the law of expansive force, as well as that of the power of cohesion by which it is counteracted, is entirely unknown, it is obvious that no satisfactory solution can be given to this question, without actual experiment."

QUESTION VI.

Solved by Charles Richards.

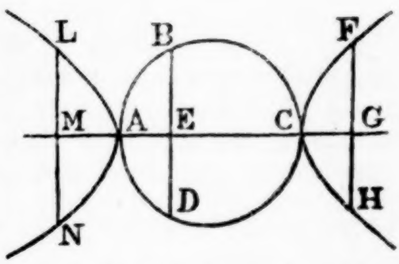
Put $d=$ the diameter of the sphere, $a=.5236$, and $x=$ the height of a segment containing one third of the sphere, this height being taken from the extremity of a diameter: then, *by mensuration*, $ad^3=$ the solidity of the sphere, and $(3d-2x)ax^2=$ the solidity of the segment having the height x, consequently $(3d-2x)ax^2=\frac{ad^3}{3}$, which divided by a becomes $(3d-2x)x^2=\frac{1}{3}d^3$: this cubic equation resolved gives $x=.386963d$, which shows the dis-

tance from each extremity of the diameter d to its intersections with the parallel planes. In the same manner we may determine the height of a segment that shall have any given ratio to the sphere.

SCHOLIUM BY THE EDITOR.

I begin with presenting my readers with the following passage from *Bossu's History of the Mathematics*. "Though few of the works of Diocles have came down to us, we have enough to inform us, that he was endowed with great sagacity. Beside his cissoid, he discovered the solution of a problem which Archimedes had proposed in his treatise on the sphere and cylinder, and which consisted in cutting a sphere by a plane in a given ratio. We know not whether Archimedes himself had resolved this question, at that time very difficult, and which leads to an equation of the third order in the modern methods. The solution of Diocles, which is learned and profound, terminates in a geometrical construction by means of two conic sections cutting each other;" to which quotation I add, that the problem may be constructed with greater simplicity by the intersections of one conic section with the circumference of a circle.

In examining with attention the algebraic method of resolving this problem, we meet with a very curious circumstance, which will probably appear somewhat mysterious to such as have not meditated profoundly on the subject. It is evident that there can be but one answer to the problem; for unless x, the height of the segment, be equal to $.386963 \times d$, the solid content of the segment cannot possibly be equal to one third part of the whole sphere; yet it is remarkable that there are two other values of x which fulfil the equation, $(3d-2x)x^2=\frac{1}{3}d^3$, one of these values being positive and the other negative! Whence come these values of x? and what do they signify? Can we cut off from d a negative height for x? or a positive height greater than the diameter? for there is a value of x greater than d that satisfies the equation, $(3d-2x)\,x^2=\frac{1}{3}d^3$. I shall endeavour to answer these curious questions.



On the given straight line AC, =unity, as a diameter, describe the circle ABCD, and the equilateral or rectangular hyperbolas FCH, LAN; and draw the ordinates FGH, BED, LMN, at right angles to MAG; the common property of these curves being that the square of any semiordinate, $\overline{BE}^2$ or $\overline{FG}^2$, is equal to the rectangle of its distances from the

vertices A and C, that is, to AE.EC, or to AG.GC. Now supposing this assemblage of curves to revolve round the straight line MG, let us examine the measures of the generated sphere and hyperbolic conoids.

Put $AE=x$, solidity of the segment $BAD=S$, that of the globe $ABCD=G$, and of $FCH=H$; also put $p=3.1416$. Then $\overline{BE}^2=x(1-x)$, and $\dot{S}=px\dot{x}(1-x)=p(x\dot{x}-x^2\dot{x})$; which fluxion is evidently positive when $1-x$ is positive, that is, while x is less than unity, and the fluent $S=p(\frac{1}{2}x^2-\frac{1}{3}x^3)$ is the true value of the solid having its height $AE=x$. But when x is greater than unity, that is, when AE becomes AG, then the formula $\dot{S}=px\dot{x}(1-x)$ is the negative value of the fluxion of the solid FCH, and the correct fluent is $S=p(\frac{1}{2}x^2-\frac{1}{3}x^3)=G-H$; so that the formula $p(\frac{1}{2}x^2-\frac{1}{3}x^3)$ does not express the sum of the solids ABCD and FCH, but their difference: it is evident therefore that the very same expression $p(\frac{1}{2}x^2-\frac{1}{3}x^3)$ which is the value of the segment ABAD of the globe, when x is less than the diameter, unity, is also the true value of the excess of the globe above the hyperbolic conoid FCH when $x=AG$ is greater than the diameter, AC. The original problem requiring us to cut off a segment from the globe which should be $=\frac{1}{3}G$ produces the equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, in which equation there must evidently be one, and only one value of x which answers the question, and is less than AC: but if the question has been, to find AG such that the excess of the globe above the conoid FCG may be equal to $\frac{1}{3}G$, we should have arrived at the very same equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, which equation must evidently have one and only one value of x that will answer the question, this value being greater than the diameter AC; we see clearly, therefore, the origin, signification, and use of the two positive roots in the equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$.

Again, if we denote AM by x, we have the fluxion of the solid $LAN=\dot{S}=p\dot{x}\times(x+x^2)=p(x\dot{x}+x^2\dot{x})$ and $S=p(\frac{1}{2}x^2+\frac{1}{3}x^3)=$ the conoid LAN; and if we suppose this conoid $=\frac{1}{3}G$, we shall have the equation $p(\frac{1}{2}x^2+\frac{1}{3}x^3)=\frac{1}{3}G$, in which it is manifest that x must have one, and only one value that will answer the question: and if in this equation we write $-x$ for $+x$ we obtain the original equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$. It is evident therefore that there are three cognate though different problems, each of which leads to the equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$, and that the three roots of this equation are the answers to the problem, with this condition, that one of these answers coming out negative must have its sign changed.

There is a certain connexion or relation among the answers of these three distinct problems, which deserves notice. Supposing any of the three solids, viz. the conoid LAN, the segment of the

globe BAD, or the excess of the globe ABCD above the conoid FCH, to be given $=\frac{1}{3}G$, and that AM, AE, or AG is required, we shall, in each case, have to resolve the cubic equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$: but if any one of these three AM, AE, or AG is known, the other two may be found by a quadratic; and if two of the three are known, the third is obtained by a simple equation. This may be completely demonstrated; but it is sufficient at present to observe, that as these three AM, AE, AG, are the values of x in the equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)=\frac{1}{3}G$; we have only to divide the equation $p(\frac{1}{2}x^2-\frac{1}{3}x^3)-\frac{1}{3}G=0$, by $x-r$, r being any known one of the three roots, and the quadratic quotient equated to 0, and resolved will give the other two roots.

I am obliged to omit several other curious remarks on this subject, and shall content myself with observing that, from what has been said, we discover the value of x that renders $\frac{1}{2}x^2-\frac{1}{3}x^3$ a maximum, in the simplest manner imaginable, x being taken positive, we have only to take x such that $p(\frac{1}{2}x^2-\frac{1}{3}x^3)$ may express the solidity of the whole globe ABCD, which is manifestly the case when $x=$ the diameter AC, =unity.

QUESTION VII.

Solved by the Editor.

To this curious question I have received no true solution. Several ingenious persons were led astray by taking it for granted that the sun's apparent diameter increases fastest, when the earth, in its elliptic orbit, advances fastest directly towards the sun; but a proper degree of attention will easily discover the fallacy of this principle.

Put $1=$ semitransverse of the earth's orbit, $c=$ semiconjugate, $e=.0168$, =the eccentricity in the year 1807; also let $d=$ the real semidiameter of the sun in parts of the semitransverse of the earth's orbit, unity, $D=$ the measure of the sun's apparent semidiameter, being an arc of a great circle to the radius unity, $\mathfrak{S}=$ the sine of D and v its cosine, $x=$ the radius vector, or the straight line extending from the centre of the sun to the centre of the earth at any point in its orbit, $y=$ the perpendicular from the centre of the sun to the tangent of the orbit at that point, $u=$ the tangent intercepted between the same point and y, and $t=$ the time.

As $\dot{t} : 1'' :: \dot{D} : \frac{\dot{D}}{\dot{t}} =$ the measure of the increment which D would uniformly acquire in one second of time, with its rate of increase at the beginning of that second, and $\frac{2\dot{D}}{\dot{t}} =$ the quantity

which by the question must be a maximum, and therefore $\frac{\dot{D}}{\dot{t}}$ must also be a maximum. Now $\dot{D}=\frac{\dot{S}}{v}$; also, by similar triangles, as $1 : s :: x : d=sx$, whence $s=\frac{d}{x}$, and $\dot{S}=-\frac{d\dot{x}}{x^2}$ therefore $\frac{\dot{D}}{\dot{t}}=-\frac{d\dot{x}}{x^2v\dot{t}}$. Again, let z be the elliptic curve described by the centre of the earth, and $a=2$ the given area described by x in one second, and we have the two equations, $a\dot{t}=y\dot{z}$, and $-\frac{\dot{x}}{\dot{z}}=\frac{u}{x}$ consequently $\frac{\dot{D}}{\dot{t}}=\frac{-ad\dot{x}}{x^2yv\dot{z}}=\frac{adu}{x^3yv}$, and therefore $\frac{u}{x^3yv}$ must be a maximum, and its square $\frac{u^2}{x^6y^2v^2}$.

Farther, by conics, $y^2=\frac{c^2x}{2-x}$; also $u^2=x^2-y^2$, and $v^2=1-s^2=\frac{x^2-d^2}{x^2}$, whence by substitution we have $\frac{u^2}{x^6y^2v^2}=\frac{2x-x^2-c^2}{c^2(x^6-d^2x^4)}$; so that $\frac{2x-x^2-c^2}{x^6-d^2x^4}$ must be a maximum, the fluxion of which equated to 0 gives us the equation, $(x^6-d^2x^4)\times(2\dot{x}-2x\dot{x})-(2x-x^2-c^2)\times(6x^5\dot{x}-4d^2x^3\dot{x})=0$; and by division $(x^3-d^2x)\times(1-x)-(2x-x^2-c^2)\times(3x^2-2d^2)=0$; which reduced produces the biquadratic equation $2x^4-5x^3+(3c^2-d^2)x^2+3d^2x-2c^2d^2=0$.

In resolving this equation we discover that the value of d^2, $=.0000218$, has no sensible influence on that of x; so that putting $d^2=0$, we have $2x^2-5x+3c^2=0$, from which we obtain $x=1-3e^2=.9991547$. Now half the latus rectum being $c^2=1-e^2$, we perceive that at the time required the earth has passed the extremity of the latus rectum in descending towards the perihelion, and that the sun's anomaly is more than 3 signs, which latter is the anomaly resulting from the erroneous principle mentioned above. In fact we find from Mayer's tables the true anomaly corresponding to the distance .9991547 to be $3^s\ 1^\circ\ 55'\ 4.8''$, and the time required in the year 1807 to be on the 5th day of October, at 19 minutes past 7 in the evening, on the meridian of Greenwich.

QUESTION VIII.

Solved by Charles Richards.

Let x, mx, and nx, be the numbers sought, and by the question we have the equations $(1+m^2)\overline{nx^3}=a$, $(1+n^2)\overline{mx^3}=b$, and $(m^2+n^2)x^3=c$; the last of which divided by each of the other two gives $\frac{m^2+n^2}{(1+m^2)n}=\frac{c}{a}=d$, and $\frac{m^2+n^2}{(1+n^2)m}=\frac{c}{b}=e$; whence $m^2+n^2=dn(1+m^2)$ and $m^2+n^2=em(1+n^2)$.

From the former of these we find $m=\left(\frac{n^2-dn}{dn-1}\right)^{\frac{1}{2}}$, which substituted for m in the latter gives $\frac{dn^3-dn}{dn-1}=e(1+n^2).\left(\frac{n^2-dn}{dn-1}\right)^{\frac{1}{2}}$, which reduced (putting $\frac{d^2+e^2+e^2d^2}{de^2}=r$, and $\frac{e^2+d^2e^2-d^2}{de^2}=s$,) becomes $n^6-rn^5+3n^4-2sn^3+3n^2-rn+1=0=n^3-rn^2+3n^2-2s+\frac{3}{n}-\frac{r}{n^2}+\frac{1}{n^3}$, and this by putting $n+\frac{1}{n}=u$, and $2r-2s=t$, becomes $u^3-ru^2+t=0$; whence $u=\frac{181}{90}$, $n=\frac{9}{10}$, $m=\frac{21}{20}$, $x=40$, $mx=42$, and $nx=36$, the numbers required.

The same Question solved by the proposer, John Capp.

Let x, y, z, be the numbers sought, and by the question we have the three equations $x^2z+y^2z=a$, $x^2y+z^2y=b$, $y^2x+z^2x=c$. Put $x+y+z=u$, $xy+xz+yz=v$, and $xyz=p$. The sum of the squares of the three given equations being taken gives us $x^4z^2+y^4z^2+x^4y^2+z^4y^2+y^4x^2+z^4x^2+6p^2=a^2+b^2+c^2=e$: Again, multiplying together the three given equations, and dividing the product by $x\ y\ z$, we have $x^4z^2+y^4z^2+x^4y^2+z^2y^2+y^4x^2+z^4x^2+2p^2=\frac{abc}{p}=\frac{f}{p}$; which taken from the preceding, leaves $4p^2=e-\frac{f}{p}$, and $ep^3-4p=f$, whence p becomes known.

Now multiply the equations $x+y+z=u$, and $xy+xz+yz=v$, and the product is $x^2y+x^2z+y^2z+z^2x+z^2y+3xyz=uv$, that is, (putting $g=a+b+c+3p$) $uv=g$.

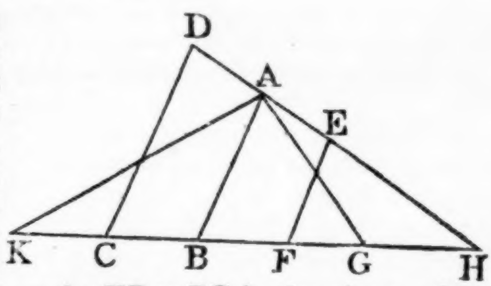
Again, squaring the equation $v=xy+xz+yz$, we have $v^2=x^2y^2+x^2z^2+y^2z^2+2x^2zy+2y^2xz+2z^2xy=x^2y^2+x^2z^2+y^2z^2+2xyz(x+y+z)=x^2y^2+x^2z^2+y^2z^2+2pu$, from which $x^2y^2+x^2z^2+y^2z^2=v^2-2pu$, which multiplied by $x^2+y^2+z^2=u^2-2v$, gives $x^4y^2+x^4z^2+y^4x^2+y^4z^2+z^4x^2+z^4y^2+3x^2y^2z^2=(v^2-2pu)\times(u^2-2v)$; that is, by what was shown above, $e-3p^2=u^2v^2-2v^3-2pu^3+4puv=g^2-2v^3-2pu^3+4pg$, whence $v^3+pu^3=$

$h=g^2+4pg+3p^2-e$: from the square of which $v^6+2pu^3v^3+p^2u^6=h^2$, taking $4pu^3v^3=4pg^3$, we have $v^6-2pu^3v^3+p^2u^6=h^2-4pg^3$, and extracting the square root, $v^3-pu^3=\sqrt{h^2-4pg}$: this last added to the equation $v^3+pu^3=h$, we have (after dividing by 2) $v^3=\frac{1}{2}(h\mp\sqrt{h^2-4pg^3})$, from which v is known, and thence u, by the equation $u=\frac{g}{v}$.

Having the values of u, v and p, we easily obtain those of x, y and z, by means of the equations $x+y+z=u$, $xy+xz+yz=v$, and $xyz=p$; from which results the equation $x^3-ux^2+vx-p=o$; and the three roots of this equation 36, 40, and 42, are the numbers required.

QUESTION IX.

Solved by Daniel Smith.



Let ABCD be the given trapezoid. Produce the oblique sides DA, CB, at pleasure, and let H be their point of intersection. Make the triangle ABK equal to ABCD, which is easily done by drawing through D a straight line DK parallel to AC. Again make KB to BG in the given ratio of ABCD to that which is to be constructed; lastly, having made HF a mean proportional between HB and HG, draw FE parallel to AB, and ABFE is the trapezoid required.

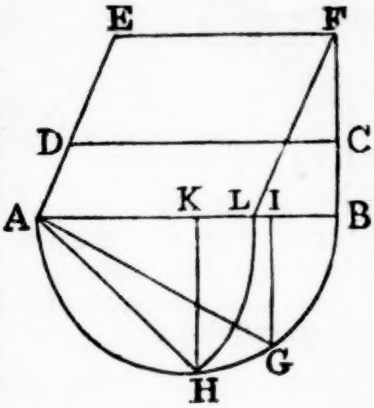
The demonstration is easy. It is plain that ABFE is equiangular to ABCD and described on one of its parallel sides. Again, since ABH is to EFH as the square of BH to the square of FH; that is, by construction, as BH is to GH, or as ABH to AGH, therefore EFH is equal to AGH, and ABFE is equal to ABG; and ABCD is to ABFE as ABK to ABG, that is, in the given ratio of KB to BG, as required.

The preceding construction determines the trapezoid required on the less parallel side of the given one: a similar construction is equally applicable for the greater parallel side.

Thus, if ABFE were the given trapezoid, of which the greater parallel side is AB, make, as before, the triangle ABG equal to ABFE, and take BG to BK in the given ratio of ABFE to the trapezoid required; finally, make the trapezoid ABCD equal to the triangle ABK, by taking CH a mean proportional between HB and HK.

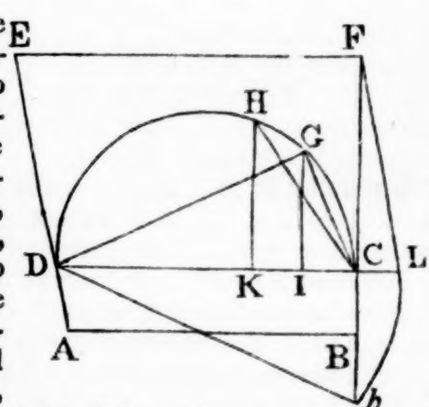
Another solution to the same, by John D. Craig.

Let ABCD be the given trapezoid, AB, DC being its parallel sides. On AB describe the semicircle AHGB, and from A apply AG equal to DC; draw GI perpendicular to AB; make IK to BI in the given ratio of the trapezoid to be applied to DC, to the given one ABCD; draw KH parallel to IG, join AH, and in AB take AL equal to AH; through L draw LF parallel to AD, and meeting BC produced in F; lastly complete the trapezoid DEFC, and the thing required is done.



For it is evident that the figures are equiangular. And AC is to DF as $AB^2—DC^2$ to $DC^2—EF^2$, (by 19. VI. E.E and division) that is, by construction, as $AB^2—AG^2$ to $AG^2—AH^3$, or as BI to IK.

If K fall on or beyond the point A, then the required construction cannot be applied to the shorter side DC. The following is general. On the greater parallel side DC describe the semicircle DHGC, in which apply DG equal to AB, and draw GI perpendicular to DC; make KC to IC in the given ratio of the given trapezoid to the arc required, and having drawn KH as before, join HC; make Ch perpendicular to DC, and equal to HC; in DC produced take DL equal to Dh, and draw LF parallel to AD as before, &c.



For $Dh^2—DC^2=Ch^2=CH^2$; but $EF=Dh$, therefore $EF^2—DC^2=HC^2$; and since $DG=AB$; $DC^2—AB^2=GC^2$: therefore $AC : DF :: GC^2 : HC^2 :: CI : CK$.

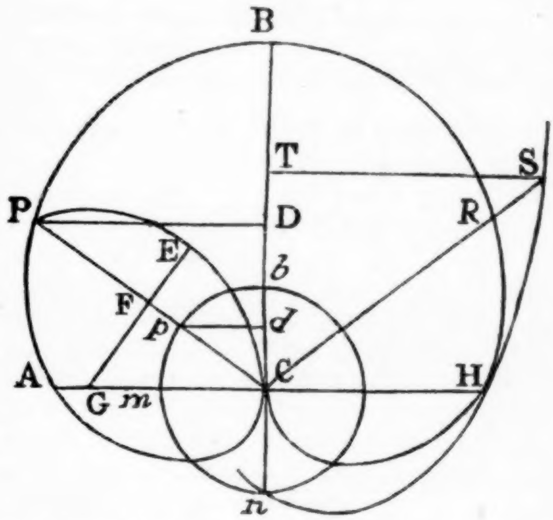
SCHOLIUM.

A much simpler construction might be given to the problem by producing the oblique sides till they would meet, (see Prob.

VIII, annexed to Mr. Thos. Simpson's Elements of Geometry:) but as the problem is of considerable use in surveying, we have endeavoured to give it an original construction.

PRIZE QUESTION.

Solved by John D. Craig.



Since the bending of the hygrometer is caused by the expansion or contraction of the transverse fibres of the wood, and since we must suppose the degree of expansion or contraction in each fibre to be the same, it follows that the degree of curvature occasioned thereby, must every where be the same also; and consequently the curve into which the hygrometer forms itself is that of the circle. The question, therefore, is to find the locus of one end of a circular arc of a given length; the other end being fixed, and its radius variable.

Let the straight line CB represent the hygrometer indicating that mean degree of moisture for which it was constructed; and by a change of moisture let it be bent into the circular arch CEP, its extremity P having in the mean time described the curve BP. Draw the chord CP; and about C as a centre, with unity for a radius, describe the circle *bpmn;* draw AC, *pd*, at right angles to CB; bisect CEP, CFP in E and F, and draw EF meeting AC in G.

Let the arch *bp*, supposed variable, be denoted by x; the ordinate CP, by y; and for CE or EP, half the length of the hygrometer, put a: then by a well known series, the sine *pd* of the arch x will be $=x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-\frac{x^7}{2.3.4.5.6.7}+$, &c. and its cosine $Cd=1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-\frac{x^6}{2.3.4.5.6}+$, &c. which for brevity we shall denote by s and c respectively.

Now by reason of the similar sections, Cbp GCE, it will be as pb : CE : : pd : CF; that is, $x : a :: s : \frac{as}{x} =$ CF; hence $y = \frac{2as}{x}$, the equation of the curve; which might be otherwise expressed in terms of the abscissa BD, and right ordinate PD, either with or without x: but these equations being less simple than the above, we shall not produce them here, but proceed to the finding of the area; for which purpose we have $\frac{y^2\dot{x}}{2} = \frac{2a^2s^2\dot{x}}{x^2}$ for the fluxion of the space BCP; whence the area corresponding to any assigned value of x will be had.

But when x exceeds a quadrantal arch, the above series (s) converges but slowly; in order therefore, to ascertain with more ease the area on the other side of AC, let $2w$ be the whole periphery of the variable circle of which $2a$ is an arch, then will $2(w-a)$ be the portion thereof intercepted between the centre C and the generating point; and retaining x to measure the angle that the revolving ordinate (or chord) makes with BC (produced towards n,) it is evident that by writing $w-a$ in place of a in the foregoing expressions, the equation of the curve, and the fluxion of its area on the other side of AC will be had, and that when P arrives at C, $w-a$ will vanish, or w become equal to a.

Again, for the rectification of the curve BP we have $\sqrt{\dot{y}^2 + y^2\dot{x}^2}$, or the fluxion of BP, $= \frac{2a}{x}\sqrt{(x\dot{s} - s\dot{x})^2 + s^2\dot{x}^2}$; the fluent of which, when $x = bm$, will give the length of AB; but to find the length of the remaining part AC, $w-a$ may be written for a as before.

It should be observed also, that the radius GC or GE is $= \frac{a}{x}$: for, by similar triangles, pd : Cp : : CF : GC; that is, as $s : 1 :: \frac{as}{x} : \frac{a}{x}$. Therefore putting p for the circumference of a circle whose radius is unity, $\frac{pa}{x}$ will, at all times, express the whole circumference of which the hygrometer is an arch; and is, therefore, always inversely as the arch bp; and thence may the value of w, for any position of the generating ordinate, be determined.

It is worthy of notice, that the greatest perpendicular PD will subtend an angle PCD, such that its tangent to the radius Cb will

be double the length of its arch bp; that is, $2x=\frac{s}{\sqrt{1-s^2}}$, answering to an angle of $66^{\circ}\ 46\frac{2}{3}'$ nearly.

Mr. *Clay*, the proposer of this prize question, also favoured the Editor with a very explicit solution, which he concludes with the following ingenious observations.

With the centre C and axis CB describe the hyperbolic spiral nHS, passing through the point H where the straight line ACH meets BHC: draw from the centre C any straight line CS to meet the curves nHS, and CHB in R and S; from S let fall ST perpendicular to the axis BC, produced either way if necessary, and CR shall be equal to ST.

When the spiral SHn passes through n, the curve in question, BHC will fall into the point C; whenever the spiral SHn crosses AH, the curve BHC also crosses CH, and whenever the spiral SHn cuts the axis BCn, the curve BHC falls into the point C: and since, as is well known, the spiral SHn performs an infinite number of gyrations about the centre C, therefore the curve BHn also performs an infinite number of gyrations, alternately cutting CH, and falling into the point C; and it will be seen, by mere inspection, that the other part of the curve CHB after passing through B will be exactly similar to the first: so that this singular curve consists of two similar infinite branches, both passing through the point C an infinite number of times, and performing an infinite number of revolutions. As it seems to differ in kind from every other known curve, I would call it *Rittenhouse's Hygrometrical curve.*

ACKNOWLEDGMENTS, &c.

A list of the names and places of residence of the ingenious gentlemen who have favoured the editor with solutions to the questions proposed in No. II. The figures annexed to the names refer to the questions answered by each as numbered in the work.

Nathaniel Bowditch, Salem, Massachusetts,		2	3	4	5	6	7	8	9	10
John Capp, Harrisburg, Pennsylvania,	1							8		
William Charington, Reading, Pennsylvania,			3			6				
William Child, Pottstown, Pennsylvania,		2								
Joseph Clay, Philadelphia,										10
Jacob Conklin, Bergen county, N. Jersey,		2								
John Coope, Philadelphia,	1	2	3	4	5	6			9	
John D. Craig, Baltimore,	1	2	3	4	5	6		8	9	10
John Eberle, Philadelphia,						6				

John Gummere, near Burlington, New-Jersey, ...	1	2	3	4	5	6			9	10
William Lenhart, Baltimore, ...								8		
Enoch Lewis, Westown, Pennsylvania,									9	
Robert Patterson, Philadelphia, ...				4	5					
Charles Richards, Reading, Pennsylvania, ...		2	3			6		8	9	
Daniel Smith, jun. Burlington, N. Jersey,	1	2	3	4	5	6			9	10
Seth Smith, Burlington, N. Jersey,	1	2	3	4	5	6			9	10
E. R. White, Danbury, Connecticut, .		2								
John Willits, near Burlington, N. Jersey,	1	2	3	4	5	6			9	

☞ The Prize of six dollars has been divided equally between Joseph Clay, the proposer, and John D. Craig, Baltimore.

ARTICLE VIII.

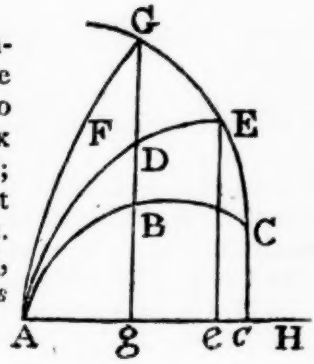
RESEARCHES CONCERNING ISOTOMOUS CURVES, &c.

BY ROBERT ADRAIN.

1. The prize question by Mr. Clay, respecting Rittenhouse's hygrometer, having recalled my former ideas on a similar inquiry, I flatter myself that I shall do an acceptable thing to geometers by presenting them with my speculations on this subject. This I do the more willingly as it will afford me an opportunity of developing a remarkable method of investigation, which not only leads us in a direct manner to the solution of multitudes of the most difficult problems, but even comprehends many curious theories as particular applications of it. I have not, however, yet examined completely the application of this method in its full extent; nor settled and arranged those analytical formulas most favourable to its improvement: in these researches I hope to be seconded by some of my fellow-labourers in the vineyard of science.

DEFINITION I.

2. Let ABC, ADE, AFG, &c. be an indefinite number of curves of the same kind, on the same plane, and referred to the same axis AH, the common vertex of all the curves being in one point A; and let CEG be such a curve as to cut off equal arcs ABC, ADE, AFG, &c. from all the former curves ABC, ADE, &c. then is CEG called an *Isotomous curve.*



In like manner, if Cc, Ee, Gg, &c. be parallel ordinates of the curves ABC, ADE, &c. referred to the axis AH; and if the curve CEG cut the curves ABC, ADE, &c. in such a manner as to make all the areas ABCc, ADEe, AFGg, &c. equal to one another; then CEG is called an *Isotomous curve*.

Or if the surfaces, or solidities, generated by all the arcs ABC, ADE, &c. or by the areas ABCc, ADEe, &c. revolving about the axis AH, be equal to one another; then also, CEG is called an *Isotomous curve*. In general, if the curve CEG cut the curves ABC, ADE, &c. in such a manner that any similar functions whatever of ABC, ADE, &c. may be equal to one another, CEG is called an *Isotomous curve*.

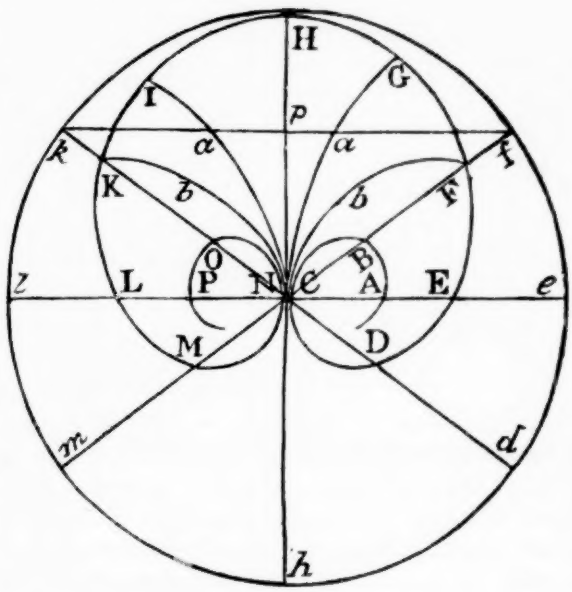
For example, if ABC, ADE, &c. be common parabolas, having the same vertex A, and axis AH; and if all the parabolic arcs ABC, ADE, &c. be equal, then CEG is the *Isotomous curve of parabolic arcs:* or if ABC, ADE, &c. be equal circular arcs, having the common vertex in A, and their centres in the axis AH; CEG is called the *Isotomous curve of circular arcs:* or if ABC, ADE, &c. be parabolas, and CEG cut off such arcs ABC, ADE, &c. that the curvature of all those parabolic arcs at their extremities C, E, G, &c. is the same; CEG is the *Isotomous curve of parabolic curvature*.

DEFINITION II.

3. Rittenhouse's hygrometer is formed of two thin pieces of wood of a uniform breadth, thickness, and length, glued together; the grain of one piece running with the length of the hygrometer, and that of the other with the breadth: a change of the degree of moisture produces an expansion or contraction in the latter piece, by which means the hygrometer evidently assumes the form of a circular arc: and when one end of the hygrometer is fixed on a plane, as well as its direction at that end, the other end describes on the plane a curve, which Mr. Clay calls *Rittenhouse's Hygrometrical curve*. In like manner, if the thickness of the hygrometer, instead of being uniform, should be supposed to vary according to any proposed law, the hygrometer will assume the form of a curve, of variable curvature; and one end being fixed as before, the other, by a change of moisture, will describe, on a plane, a line which we may, in general, call a *Hygrometrical curve*.

4. If instead of wood we use metallic laminæ of different kinds, as of brass and steel, a change of temperature will have a similar effect on the compound metallic strip with that produced on the strip of wood by a change of moisture; the same curves therefore may also be denominated *Hygrometrical*, or *Thermometrical curves*.

5. It is evident that all hygrometrical curves are also isotomous, because the variation of the length of the hygrometer, by a change of moisture, is insensible; but are all isotomous curves also hygrometrical? This is a profound question, which cannot fail to draw the attention of geometers, and is a striking example of the new and sublime researches which every where abound in this interesting department of geometry. I shall confine myself at this time to an exposition of the nature, quadrature, and rectification of the *Isotomous curve of circular arcs*, which is evidently the same with *Rittenhouse's Hygrometrical curve.*



6. Let CbF, CaG, CaI, CbK, &c. be an infinite number of circular arcs in the same plane, each being equal to the straight line CH, and touching it in the point C; or, which is the same thing, having their centres in the straight line lCe which is perpendicular to HC; and let A BCDEFGHIKL MNOP, &c. be the curve passing through the extremities F, G, I, K, &c. of all those equal arcs; the present subject of inquiry is concerning the nature and mensuration of this curve.

7. It is evident that the curve in question has two equal and similar branches, one on each side of the straight line HCh: let us trace the progress of either of these branches, beginning at the point H. If we imagine the centre of one of these equal arcs to be infinitely remote in the straight line Ce, produced indefinitely from C towards e, that arc will be coincident with the straight line CH, and as the centre of the circle to which this constant arc belongs approaches towards C, the curvature of the arc increases continually, the arc itself occupying a greater and greater portion of the whole circumference; from which it follows, that the curve HGF continually draws nearer to the centre C until it falls into it; when the centre of the arc falls in the middle of CE, the arc be-

comes a semicircle on that diameter, and when the diameter of the arc is the half of CE, the arc becomes a whole circumference.

As the centre of the arc approaches nearer to C than by a distance equal to one-fourth part of CE, the arc becomes more than a circumference, and of course its moveable extremity proceeds from the centre C, and describes an additional part of the curve in question, until the centre of the arc is distant one-eighth of CE from C, in which case the extremity of the constant arc falls again into C, from which, as the centre of the arc approaches still nearer to C, it again sets out from C to generate another orbicular portion of the proposed curve. In this manner the constant arc folds itself successively into one, two, three, four circumferences, and so on in infinitum; from which it necessarily follows, that the curve HFDCBA makes an infinite number of revolutions, passing once every revolution through the point C: and the other branch HIKL, &c. being similar and equal to the former, makes also an infinite number of revolutions on the other side of the straight line HCh, and passes once in each revolution through the same point C.

8. Let us now investigate the equation of the curve. For this purpose, let us describe the circle Hehl with the radius CH: and supposing CbF, CbK two similar arcs, let their chords be produced to meet the circumference in f and k; and the straight line fk will manifestly cut Hh at right angles in p. Now because fk is parallel to el, the angles eCf, Cfp are equal, and therefore their complements, which are the angles FCb, kfH, of the segments CbF, fHk, are also equal, consequently the segments CbF, fHk are similar, and also the arcs CbF, fHk, from which it is plain that the arc fHk is to its chord fk, or, the arc Hf to its sine fp, as the given arc CbF is to its chord CF. Put now the radius CH= the constant arc CbF= unity, the chord CF$=x$, which is the polar ordinate of the curve HFE referred to the pole or centre C; also put the corresponding arc H$f=z$, and its sine $fp=s$, and by what has just been shown, we have $z : s :: 1 : x$, and $x=\frac{s}{z}$, which is the equation of the curve.

9. This equation applies with equal propriety to all the portions of the curve HFDCB, &c. When $z=$ a quadrantal arc H$e=q$, then $s=1$, and CE$=x=\frac{1}{q}$; when $z=$ the semi-circumference H$eh=p$, then $s=o$, and $x=\frac{o}{p}=o$, and the curve passes through C; when $z=$Hehm, then CB$=x=\frac{s}{z}$; but in this case the sine s

of z, (which is negative when z terminates between h and H, on the semi-circumference hlH) must be taken positively. When z=Hehl=$3q$, then CA$=x=\frac{1}{3q}$; and when $z=$ the whole circumference HehlH, we have $x=\frac{o}{4q}=o$, and the curve DCBA falls into the centre the second time; and when z is more than one circumference, the curve goes on to make another revolution.

10. Having obtained the equation of the curve we proceed to its quadrature. By the common rules of fluxions, which we need not at present stop to explain, we find $\dot{A}=\frac{x^2\dot{z}}{2}=\frac{s^2\dot{z}}{2z^2}$, putting A for the area HCF generated by the ordinate CF. We have now to find the fluent of $\frac{s^2\dot{z}}{z^2}$ which gives the value of 2A, but this is a task which I am unable to perform without series, or some method of approximation. By expressing s in terms of z, we may reduce $\frac{s^2\dot{z}}{z^2}$ to a series of monomial terms, of which the fluents may be found, and consequently the area will be obtained which is generated while z becomes Heh; but this area may be obtained with much more facility by the method of equidistant ordinates.

As $\frac{s}{z}$ is the value of x with a negative sign when z terminates in hlH; we might be led to doubt whether the fluent of $\frac{s^2\dot{z}}{z^2}$ would also give the area generated by the ordinate CB, this area being considered as a continuation of the former; because we know that a negative ordinate in reference to an axis produces a negative area, and the analytical expression in many such cases by no means gives the true area; but in the present case all the elementary particles of the area expressed, each by $\frac{s^2\dot{z}}{z^2}$, must necessarily have the same sign, since it is only the square of s which enters into the formula; and therefore the fluent of $\frac{s^2\dot{z}}{2z^2}$ will be the sum of all the areas generated by any number whatever of ordinates CF, CB, &c. while z becomes equal to any number of circumferences.

11. But though by transforming $\frac{s^2\dot{z}}{z^2}$ into terms of z, and in-

tegrating, we should have a true value of the area of the curve, yet we could not determine the numerical value of the area from this series when z consisted of several circumferences, because the series converges too slowly. In fact, the fundamental series, on which the series for the area depends, viz. $s=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-\frac{z^7}{2.3.4.5.6.7}+$ &c. though universally true, whatever number of circumferences z may consist of, is but ill calculated to give the numerical value of s when z becomes equal to two or more circumferences: for, besides its slow degree of convergency as we advance in the series, it is subject to another considerable defect, by first diverging for a certain number of terms, before we arrive at those terms which afterwards converge in infinitum.

The series $z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2.3.4.5}=\frac{z^7}{1.2.3.4.5.6.7}$, &c. begins to show diverging terms as soon as z exceeds the value $\surd 6$, which gives an arc of $140^0\ 20'\ 43''\ 31'''$.

From what has been said, it appears evident, that the area of the curve generated by its ordinate whilst z is increased to any number of circumferences, cannot be obtained numerically by reducing the formula $\frac{s^2\dot{z}}{z^2}$ into series in the usual way; and even the method of equidistant ordinates would in such cases become immensely laborious. We must therefore search another method of investigation of greater facility.

12. We have shown that when Hf becomes Hehm, the ordinate CF becomes CB: and supposing z to denote Hf, we have H$ehm=p+z$; in this case the equation $\mathrm{CF}=x=\frac{s}{z}$ becomes $\mathrm{CB}=\frac{s}{p+z}$, and the fluxion of the area generated by the ordinate CB is therefore $\frac{\frac{1}{2}s^2\dot{z}}{(p+z)^2}$, the fluent of which gives the area inclosed by the curve CBA.

Again, when Hf becomes equal to HehlHf, then z becomes $2p+z$ and $\frac{s}{2p+z}$ is the ordinate from C along CB for the next revolution of the curve CBA, and the fluxion of this area is $\frac{\frac{1}{2}s^2\dot{z}}{(2p+z)^2}$. In like manner the next ordinate from C, in the same direction CBF, is $\frac{s}{3p+z}$, and the fluxion of the area is $\frac{\frac{1}{2}s^2\dot{z}}{(3p+z)^2}$.

of z, (which is negative when z terminates between h and H, on the semi-circumference hlH) must be taken positively. When z=Hehl=$3q$, then CA=$x=\frac{1}{3q}$; and when $z=$ the whole circumference HehlH, we have $x=\frac{o}{4q}=o$, and the curve DCBA falls into the centre the second time; and when z is more than one circumference, the curve goes on to make another revolution.

10. Having obtained the equation of the curve we proceed to its quadrature. By the common rules of fluxions, which we need not at present stop to explain, we find $\dot{A}=\frac{x^2\dot{z}}{2}=\frac{s^2\dot{z}}{2z^2}$, putting A for the area HCF generated by the ordinate CF. We have now to find the fluent of $\frac{s^2\dot{z}}{z^2}$ which gives the value of 2A, but this is a task which I am unable to perform without series, or some method of approximation. By expressing s in terms of z, we may reduce $\frac{s^2\dot{z}}{z^2}$ to a series of monomial terms, of which the fluents may be found, and consequently the area will be obtained which is generated while z becomes Heh; but this area may be obtained with much more facility by the method of equidistant ordinates.

As $\frac{s}{z}$ is the value of x with a negative sign when z terminates in hlH; we might be led to doubt whether the fluent of $\frac{s^2\dot{z}}{z^2}$ would also give the area generated by the ordinate CB, this area being considered as a continuation of the former; because we know that a negative ordinate in reference to an axis produces a negative area, and the analytical expression in many such cases by no means gives the true area; but in the present case all the elementary particles of the area expressed, each by $\frac{s^2\dot{z}}{z^2}$, must necessarily have the same sign, since it is only the square of s which enters into the formula; and therefore the fluent of $\frac{s^2\dot{z}}{2z^2}$ will be the sum of all the areas generated by any number whatever of ordinates CF, CB, &c. while z becomes equal to any number of circumferences.

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tegrating, we should have a true value of the area of the curve, yet we could not determine the numerical value of the area from this series when z consisted of several circumferences, because the series converges too slowly. In fact, the fundamental series, on which the series for the area depends, viz. $s=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-\frac{z^7}{2.3.4.5.6.7}+$ &c. though universally true, whatever number of circumferences z may consist of, is but ill calculated to give the numerical value of s when z becomes equal to two or more circumferences: for, besides its slow degree of convergency as we advance in the series, it is subject to another considerable defect, by first diverging for a certain number of terms, before we arrive at those terms which afterwards converge in infinitum.

The series $z-\frac{z^3}{1.2.3}+\frac{z^5}{1.2.3.4.5}=\frac{z^7}{1.2.3.4.5.6.7}$, &c. begins to show diverging terms as soon as z exceeds the value $\sqrt{6}$, which gives an arc of $140^0\ 20'\ 43''\ 31'''$.

From what has been said, it appears evident, that the area of the curve generated by its ordinate whilst z is increased to any number of circumferences, cannot be obtained numerically by reducing the formula $\frac{s^2\dot{z}}{z^2}$ into series in the usual way; and even the method of equidistant ordinates would in such cases become immensely laborious. We must therefore search another method of investigation of greater facility.

12. We have shown that when Hf becomes Hehm, the ordinate CF becomes CB: and supposing z to denote Hf, we have Hehm=$p+z$; in this case the equation CF$=x=\frac{s}{z}$ becomes CB $=\frac{s}{p+z}$, and the fluxion of the area generated by the ordinate CB is therefore $\frac{\frac{1}{2}s^2\dot{z}}{(p+z)^2}$, the fluent of which gives the area inclosed by the curve CBA.

Again, when Hf becomes equal to HehlHf, then z becomes $2p+z$ and $\frac{s}{2p+z}$ is the ordinate from C along CB for the next revolution of the curve CBA, and the fluxion of this area is $\frac{\frac{1}{2}s^2\dot{z}}{(2p+z)^2}$. In like manner the next ordinate from C, in the same direction CBF, is $\frac{s}{3p+z}$, and the fluxion of the area is $\frac{\frac{1}{2}s^2\dot{z}}{(3p+z)^2}$.

It is manifest therefore the sum of all the arcs generated by all the ordinates CF, CB, &c. or, which is the same thing, the whole area generated by the ordinate CF, in making an infinite number of revolutions, is equal to the sum of the fluents of $\frac{\frac{1}{2}s^2\dot{z}}{z^2} + \frac{\frac{1}{2}s^2\dot{z}}{(p+z)^2} + \frac{\frac{1}{2}s^2\dot{z}}{(2p+z)^2} + \frac{\frac{1}{2}s^2\dot{z}}{(3p+z)^2}$, &c. sine fine.

Any one of these areas may be determined separately; but the whole sum of any proposed number of them may be found thus; since all these areas have the common element $\dot{z}$, it is evident that their sum will be equal to the fluent of $\frac{1}{2}\dot{z} \times \left\{\frac{s^2}{z^2} + \frac{s^2}{(p+z)^2} + \frac{s^2}{(2p+z)^2} + \frac{s^2}{(3p+z)^2}, \text{ \&c.}\right\}$

Hence we obtain a much simpler method of obtaining any number of these areas: we have only to divide the semi-circumference Heh into a sufficient number of equal parts, and complete the several corresponding values of $\frac{1}{2}\left\{\frac{s^2}{z^2} + \frac{s^2}{(p+z)^2} + \frac{s^2}{(2p+z)^2} \text{ \&c.}\right\}$ as so many perpendicular ordinates, and the known rules of equidistant ordinates will give the area sought.

13. But the total area of the curve, or the sum of all the areas generated by the revolving ordinate CF in an infinite number of gyrations, may be found by a series of given finite quantities in the following manner, which is remarkably curious.

The whole sum of these areas on both sides of CH being the whole fluent of $\frac{s^2\dot{z}}{z^2}$, together with the whole fluent of $s^2\dot{z}$ $\left\{\frac{1}{(p+z)^2} + \frac{1}{(2p+z)^2} + \frac{1}{(3p+z)^2}, \text{ \&c. sine fine}\right\}$; let us put the former fluent S, which is the area bounded by the curve CDEF GHIKLMN till its two branches first fall into the pole C; and in order to obtain the latter fluent, let each of the terms $\frac{1}{(p+z)^2}$, $\frac{1}{(2p+z)^2}$, &c. be reduced to series, and we shall have

$$\frac{1}{(p+z)^2}=\frac{1}{p^2}-\frac{2z}{p^3}+\frac{3z^2}{p^4}-\frac{4z^3}{p^5}+\frac{5z^4}{p^6}\ \&c.$$
$$\frac{1}{(2p+z)^2}=\frac{1}{2^2.p^2}-\frac{2z}{2^3.p^3}+\frac{3z^2}{2^4.p^4}-\frac{4z^3}{2^5.p^5}+\frac{5z^4}{2^6.p^6}\ \&c.$$
$$\frac{1}{(3p+z)^2}=\frac{1}{3^2.p^2}-\frac{2z}{3^3.p^3}+\frac{3z^2}{3^4.p^4}-\frac{4z^3}{3^5.p^5}+\frac{5z^4}{3^6.p^6}\ \&c.$$
$$\frac{1}{(4p+z)^2}=\frac{1}{4^2.p^2}-\frac{2z}{4^3.p^3}+\frac{3z^2}{4^4.p^4}\quad \&c.\quad \&c.$$
$$\frac{1}{(5p+z)^2}=\frac{1}{5^2.p^2}-\frac{2z}{5^3.p^3}\quad \&c.\quad \&c.$$
$$\&c. = \&c.\quad \&c.$$

Put now $a=\frac{1}{p^2}\times:1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}$ &c.

$$b=\frac{1}{p^3}\times:1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}\ \&c.$$
$$c=\frac{1}{p^4}\times:1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}\ \&c.$$

and the sum of all the preceding series for the value of $\frac{1}{(p+z)^2}+\frac{1}{(2p+z)^2}+$ &c. will become $a-2bz+3cz^2-4dz^3+5ez^4$, &c. The series expressing the values of a, b, c, d, &c. are all summable, as has been shown by the great mathematicians John Bernouilli, and John Landen; the values of these letters a, b, &c. are in fact common rational fractions.

We have now converted $s^2\dot{z}\left\{\frac{1}{(p+z)^2}+\frac{1}{(2p+z)^2}+\&c.\right\}$ into $s^2\dot{z}\times\left\{a-2bz+3cz^2-4dz^3+5ez^4, \&c.\right\}$; from which the area required is readily determined, by substituting for s its value $z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-$ &c. in terms of z, then squaring this series, and multiplying, we shall have the fluxion of the area in terms of z: lastly, the fluent being taken, and p substituted for z, we shall have a series in terms of p, and the sum of this series P united to S gives P+S for the whole area sought.

14. We may however proceed in a very different manner to investigate the fluent of $s^2\dot{z}\times\left\{a-bz+cz^2, \&c.\right\}$ Put $\dot{A}=s^2\dot{z}$, $\dot{B}=s^2z\dot{z}$, $\dot{C}=s^2z^2\dot{z}$, $\dot{D}=s^2z^3\dot{z}$, &c. and we have $A=\frac{z}{2}-\frac{sy}{2}$, (put-

ting $y=$ the cosine of z,) which, when $z=p$, becomes $\frac{1}{2}p$. This is easily found, for $s^2\dot{z}=\frac{s^2\dot{z}}{\sqrt{1-s^2}}$, the fluent of which is obtained by the common rules given by authors on fluxions.

Again, $B=\frac{1}{4}z^2+\frac{1}{4}s^2-\frac{1}{2}syz$, which becomes $B=\frac{1}{4}p^2$, when z becomes equal to p.

$C=\frac{z^3}{6}-\frac{z}{4}+\frac{1}{2}s^2z-\frac{1}{2}syz^2+\frac{1}{4}xy$, which, when $z=p$, becomes $C=\frac{1}{6}p^3-\frac{1}{4}p$.

$D=\frac{1}{8}p^4-\frac{3}{8}p^2$, when $z=p$.

$E=\frac{1}{10}p^5-\frac{1}{2}p^3+\frac{3}{4}p$, when $z=p$, &c.

All these fluents are obtained from the method of integration by parts, which is familiarly known to mathematicians.

Having thus obtained the values of A, B, C, &c. we have the whole fluent of $s^2\dot{z}\left\{a-bz+cz^2, \&c.\right\}=aA-2bB+3cC+4dD$ —, &c. which expresses the value of the area P.

15. We are now to examine the length of the curve.

From the original equation, $x=\frac{s}{z}$, we may easily obtain the fluxion of the curve HF, by proceeding according to the common rules given by the writers on fluxions for the rectification of curves; but by following this method we are subject to the same difficulties which occurred with respect to the quadrature; because the series which we find, are of too slow convergency to be of any use, when z becomes equal to two or three circumferences.

This difficulty is obviated by the same method which we applied to find the area; we are to determine the fluent of the fluxion of the curve by series, or otherwise, from $z=o$ to $z=p$. The next revolution of the curve must be treated in a similar manner, by using $x=\frac{s}{p+z}$ for the fluxion of the curve; the next by using $x=\frac{s}{2p+z}$; and so on for any number of revolutions: in this manner any required portions of the length of the curve may be determined.

16. Instead of engaging in these calculations, which are very complex and laborious, we shall demonstrate in an easy manner, *that the area of the curve is finite, but that the length of the curve is infinite.*

17. When the extremity F of the constant arc CbF passes successively through C, this arc folds itself successively into one, two, three, four, &c. circumferences. The length of the first of these circumferences is evidently the same with the length of the arc

CbF, = unity; the length of the second circumference is $\frac{1}{2}$, because in this case the arc CbF is twofold; the length of the third circumference is $\frac{1}{3}$, because the arc CbF then becomes threefold, the fourth circumference is $\frac{1}{4}$, and so on: and therefore the sum of all these circumferences continued indefinitely is equal to the sum of the infinite series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$, &c. and it is well known to mathematicians that the sum of this series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. sine fine, is infinite.

Now it is manifest, that the arc HFEDC of the curve is greater than the circumference of the first circle, or its equal CH; and the next orbicular arc of the curve CBAC is greater than the second circumference $\frac{1}{2}$, because the curve CBAC falls wholly without this second circumference, and any line that surrounds a circle is greater than the circumference of the circle. In like manner the third orbicular arc of the curve is greater than the next inclosed circumference $\frac{1}{3}$; and therefore the sum of all the orbicular arcs of the curve is greater than the sum of all those circumferences; that is, the whole length of the curve is greater than the sum of the infinite series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$, &c. and is therefore infinite.

18. Let a be the area of a circle of which the circumference is unity; and the area of a circle of which the circumference is $\frac{1}{2}$, will be $a\times\frac{1}{4}$; and if the circumference be $\frac{1}{3}$ the area will be $a\times\frac{1}{9}$; and therefore the areas of all the circles of which the circumferences are 1, $\frac{1}{2}$, $\frac{1}{3}$, &c. will be equal to the sum of the infinite series $a\times : 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+$, &c. The sum of this elegant series was first discovered by John Bernouilli, who showed that the sum of the infinite series $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}$ &c. is precisely $\frac{p^2}{6}$, p being the semicircumference to the radius unity: from which it follows that the sum of the areas of all the circles under consideration is $\frac{ap^2}{6}$.

Now the first of these circles is evidently greater than the first orbicular area of the curve contained within it; the second circle is also greater than the second area of the curve, and so on; and therefore the sum of all the circles, viz. $\frac{ap^2}{6}$, is greater than the sum of all the areas generated by the ordinate CF in making an infinite number of revolutions, exclusive of the first area contained by the curve HFDC and the straight line CH; from which it follows that the whole area of the curve is finite.

The principles on which these arguments are founded would also lead us to some easy approximations both of the area, and length of the curve, but our limits will not permit us to enlarge farther on the subject at present.

To be continued.

ARTICLE IX.

The Editor having received no satisfactory solution to the 20th question in the Analyst, judges it proper to have the question re-proposed.

QUESTION,

By Robert Patterson, Philadelphia.

For the best satisfactory solution of which, to be adjudged by the Editor, he offers a prize of ten dollars.

In order to find the content of a piece of ground, having a plane level surface, I measured, with a common circumferentor and chain, the bearings and lengths of its several sides, or boundary lines, which I found as follows:

1. N. 45° E. 40 perches,
2. S. 30° W. 25 ditto,
3. S. 5° E. 36 ditto,
4. West 29.6 ditto,
5. N. 20° E. 31. ditto, to the place of beginning.

But, upon casting up the difference of latitude and departure, I discovered, what will perhaps always be the case in actual surveys, that some error had been contracted in taking the dimensions. Now it is required to compute the area of this enclosure, on the *most probable supposition* of this error.

John Garnett, of New-Brunswick, New-Jersey,

offers a prize of six dollars, to be awarded by the Editor, for the best and most accurate solution to the following question:

The sun's right ascension at noon, and moon's right ascension at noon and midnight, being always given in the Nautical Almanaç for Greenwich; required, on any day, the time when the moon's centre will be on the meridian of any place whose longitude from Greenwich is known: For example, at Philadelphia, the 20th day of August, 1808. Philadelphia being supposed $5^h\ 0^m\ 55^s$ west from Greenwich.

PRIZE QUESTION.

Robert Patterson, of Philadelphia, offers a prize of six dollars, to be awarded by the Editor, at any time he may think proper, not more than five months after the publication of this number, to the author of the most simple and accurate method of finding the variation of the magnetic needle on land; without the aid of any other instrument except the common surveying-compass, and a watch that will keep time within five minutes in the week: the greatest error in variation not to exceed five minutes of a degree. The method must, of course, include directions for correcting the watch within the necessary degree of accuracy.

NEW PROBLEM,

By Robert Adrain.

It is required to investigate the nature of the Elastic Oval, or the figure which a perfectly elastic circular hoop, of uniform strength and thickness, will assume, when acted on by two equal and opposite forces at the extremities of a diameter.

ARTICLE X.

NEW QUESTIONS,

TO BE SOLVED IN THE NEXT NUMBER.

I. QUESTION 21.

By the Editor.

Supposing a hundred mathematicians, all dwelling in different towns of the United States, and no three of them in the same right line; how many roads will be necessary that each may pay a visit by a direct route to any of the rest?

II. QUESTION 22.

By the same.

If the circumference of a circle be divided into 360 equal parts by straight lines drawn from the centre; how many angles will thus be formed at the centre? each angle being supposed less than two right angles.

III. QUESTION 23.

By Robert Patterson, Philadelphia.

Two brothers, by the death of their father, became joint heirs of a tract of land in form of a rectangle or oblong, of which the

length was 300 perches, and the breadth 200. At the distance of 250 perches from one corner, and of 170 from the opposite corner, there was, within the tract, a fine spring of water, through which they agreed that the partition line should pass. The position, and length of this line, when the shortest possible, are required; so that the elder brother (as the law directed) may have two parts, and the younger brother one.

IV. QUESTION 24.

By the same.

It was observed that a cylindrical vessel whose length was double the diameter of the bottom, and open at the top, contained exactly ten gallons more when full, than when inclined in an angle of 40°; the capacity of this vessel is required.

V. QUESTION 25.

By John Forsyth, Yorktown, Pennsylvania.

A man takes out a right for 400 acres of land, and has it laid off for him in the form of an oblong; one side being 400 perches, and the other 160. Some time afterwards he applies for a preemption right of 1000 acres, to be laid off around his original tract, in such a manner that the four straight boundaries of the whole may be parallel to, and equidistant from, the four boundaries of the original survey.

VI. QUESTION 26.

By John Gummere, near Burlington, New-Jersey.

In a plane triangle, there are given one of the angles $61^\circ\ 30'$, the area 455, and the radius of the circumscribing circle 20; required the sides.

VII. QUESTION 27.

By John Hasler, West-Point, New-York State.

To find the content of a right-angled triangle having given the hypothenuse;—the rectangle of the other two sides being equal to the square of their difference.

VIII. QUESTION 28.

By William Lenhart, Baltimore.

If on the given base of a plane triangle, between two acute angles, a semicircle be described, the circumference of which

cuts the other two sides; and there be given the ratios of the segments of each side: It is required to determine the triangle.

IX. QUESTION 29.

By Charles Richards, Reading.

Given the base of a right-angled triangle=10, and the rectangle of the hypothenuse and perpendicular =200; to determine the triangle.

X. PRIZE QUESTION 30.

By John Coope, Philadelphia.

Given the sides of a trapezium, viz. AB=6, BC=12, CD=8, and DA=10; also the sum of the opposite angles ABC, ADC= 300°: required the area of the trapezium.

☞ The Editor has received, from Nathaniel Bowditch, of Salem, Massachusetts, an ingenious solution to R. Patterson's prize question, respecting the correction of a survey. It came to hand, however, too late for insertion in the present number of the Analyst, but shall appear in the next.

RULES TO BE OBSERVED BY CONTRIBUTORS.

1. All communications must be post paid and directed to Robert Adrain, Editor of the Analyst, Reading, Pennsylvania.

2. Those who wish to have new questions inserted must send a true solution along with each question.

3. Any mathematical question sent to the Editor, with or without a solution, and accompanied by a prize of not less than six dollars in value, shall, if judged admissible, be published as a prize question, with the proposer's name; and the prize shall be awarded, by the Editor, to the author of the best satisfactory solution. But if no such solution shall be furnished within five months after publication, the prize may then be appropriated, by the donor or the Editor, to some other suitable question.

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